

# Biconservative submanifolds in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$

F. Manfio, N. C. Turgay and A. Upadhyay

## Abstract

In this paper we study biconservative submanifolds in  $\mathbb{S}^n \times \mathbb{R}$  and  $\mathbb{H}^n \times \mathbb{R}$  with parallel mean curvature vector field and co-dimension 2. We obtain some necessary and sufficient conditions for such submanifolds to be conservative. In particular, we obtain a complete classification of 3-dimensional biconservative submanifolds in  $\mathbb{S}^4 \times \mathbb{R}$  and  $\mathbb{H}^4 \times \mathbb{R}$  with nonzero parallel mean curvature vector field. We also get some results for biharmonic submanifolds in  $\mathbb{S}^n \times \mathbb{R}$  and  $\mathbb{H}^n \times \mathbb{R}$ .

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## 1 Introduction

Roughly speaking, *biconservative* submanifolds arise as the vanishing of the stress-energy tensor associated to the variational problem of biharmonic submanifolds. More precisely, an isometric immersion  $f : M \rightarrow N$  between two Riemannian manifolds is biconservative if the tangent component of its bitension field is identically zero (see Section 2).

Simplest examples of biconservative hypersurfaces in space forms are those that have constant mean curvature. In this case, the condition of biconservative becomes  $2A(\text{grad } H) + H \text{grad } H = 0$ , where  $A$  is the shape operator and  $H$  is the mean curvature function of the hypersurface. The case of surfaces in  $\mathbb{R}^3$  was considered by Hasanis-Vlachos [11], and surfaces in  $\mathbb{S}^3$  and  $\mathbb{H}^3$  was studied by Caddeo-Montaldo-Oniciuc-Piu [2]. In the Euclidean space  $\mathbb{R}^3$ , these surfaces are rotational. Recent results in the study of biconservative submanifolds were obtained, for example, in [8–10, 19, 20, 22, 23].

Apart from space forms, however, there are few Riemannian manifolds for which biconservative submanifolds are classified. Recently, this was considered for surfaces with parallel mean curvature vector field in  $\mathbb{S}^n \times \mathbb{R}$  and  $\mathbb{H}^n \times \mathbb{R}$  in [7], where they found explicit parametrizations for such submanifolds.

In this paper, we give a complete classification of biconservative submanifolds in  $\mathbb{Q}_\epsilon^4 \times \mathbb{R}$  with nonzero parallel mean curvature vector field and

co-dimension 2. This extends the one obtained in [7]. To state our result, let  $\mathbb{Q}_\epsilon^n$  denote either the unit sphere  $\mathbb{S}^n$  or the hyperbolic space  $\mathbb{H}^n$ , according as  $\epsilon = 1$  or  $\epsilon = -1$ , respectively. Given an isometric immersion  $f : M^m \rightarrow \mathbb{Q}_\epsilon^n \times \mathbb{R}$ , let  $\partial_t$  be a unit vector field tangent to the second factor. Then, a tangent vector field  $T$  on  $M^m$  and a normal vector field  $\eta$  along  $f$  are defined by

$$\partial_t = f_*T + \eta. \quad (1.1)$$

Consider now an oriented minimal surface  $\phi : M^2 \rightarrow \mathbb{Q}_a^2 \times \mathbb{R}$  such that the vector field  $T$  defined by (1.1) is nowhere vanishing, where  $a \neq 0$  and  $|a| < 1$ . Let  $b > 0$  be a real number such that  $a^2 + b^2 = 1$ . Let now

$$f : M^3 = M^2 \times I \rightarrow \mathbb{Q}_\epsilon^4 \times \mathbb{R}$$

be given by

$$f(p, s) = \left( b \cos \frac{s}{b}, b \sin \frac{s}{b}, \phi(p) \right). \quad (1.2)$$

**Theorem 1.1.** *The map  $f$  defines, at regular points, an isometric immersion with  $\langle H, \eta \rangle = 0$ , where  $H$  is the mean curvature vector field of  $f$ . Moreover,  $f$  is a biconservative isometric immersion with parallel mean curvature vector field if and only if  $\phi$  is a vertical cylinder. Conversely, any biconservative isometric immersion  $f : M^3 \rightarrow \mathbb{Q}_\epsilon^4 \times \mathbb{R}$  with nonzero parallel mean curvature vector field, such that the vector field  $T$  defined by (1.1) is nowhere vanishing, is locally given in this way.*

In particular, we prove (see Corollary 5.2) that the submanifolds of Theorem 1.1 belong to a special class, which consists of isometric immersions  $f : M^m \rightarrow \mathbb{Q}_\epsilon^n \times \mathbb{R}$  with the property that the vector field  $T$  is an eigenvector of all shape operators of  $f$ .

The paper is organized as follows. In Section 2, we recall some properties of biharmonic maps and we give a more precise statement of biconservative submanifolds. The basics of submanifolds theory in product space is discussed in Section 3. In particular, we recall with details the class  $\mathcal{A}$ . In Section 4 we show some general results about  $n$ -dimensional biconservative submanifolds in  $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ . In particular, we obtain a necessary and sufficient condition for a biconservative submanifold with parallel mean curvature vector to be biharmonic. Finally, Section 5 contains the arguments necessary to prove the above main Theorem.

## 2 Preliminaries

Given a smooth map  $f : M \rightarrow N$  between two Riemannian manifolds, the energy density of  $f$  is the smooth function  $e(f) : M \rightarrow \mathbb{R}$  defined by

$$e(f) = \frac{1}{2} \|df\|^2,$$

where  $\|df\|^2$  denotes the Hilbert-Schmidt norm of  $df$ . The total energy of  $f$ , denoted by  $E(f)$ , is given by integrating the energy density over  $M$ ,

$$E(f) = \frac{1}{2} \int_M \|df\|^2 dM.$$

The map  $f$  is called *harmonic* if it is a critical point of the energy functional  $E$ . Equivalently,  $f$  is harmonic if it satisfies the Euler-Lagrange equation  $\tau(f) = 0$ , where

$$\tau(f) = \text{trace}(\nabla df)$$

is known as the tension field of map  $f$ . When  $f : M^m \rightarrow N^n$  is an isometric immersion with mean curvature vector field  $H$ , we have  $\tau(f) = mH$ . Therefore the immersion  $f$  is a harmonic map if and only if  $M$  is a minimal submanifold of  $N$ .

A natural generalization of harmonic maps are the *biharmonic* maps, which are critical points of the bienergy functional

$$E_2(f) = \frac{1}{2} \int_M \|\tau(f)\|^2 dM.$$

This generalization, initially suggested by Eells-Sampson [6], was studied by Jiang [13], where he derived the corresponding Euler-Lagrange equation

$$\tau_2(f) = J(\tau(f)) = 0,$$

where  $J(\tau(f)) = \Delta\tau(f) - \text{trace} \tilde{R}(df, \tau(f))df$  is the Jacobi operator of  $f$ .

When  $f : M^n \rightarrow N^n$  is an isometric immersion, we get

$$\tau_2(f) = m(\Delta H - \tilde{R}(df, H)df).$$

Thus a minimal isometric immersion in the Euclidean space is trivially biharmonic. Concerning biharmonic submanifolds in the Euclidean space, one of the main problems is the following known Chen's conjecture [4]: *Any biharmonic submanifold in the Euclidean space is minimal.*

The *stress-energy* tensor, described by Hilbert [12], is a symmetric 2-covariant tensor  $S$  associated to a variational problem that is conservative at the critical points. Such tensor was employed by Baird-Eells [1] in the study of harmonic maps. In this context, it is given by

$$S = \frac{1}{2} \|df\|^2 \langle \cdot, \cdot \rangle_M - f^* \langle \cdot, \cdot \rangle_N,$$

and it satisfies

$$\text{div} S = -\langle \tau(f), df \rangle.$$

Therefore,  $\text{div} S = 0$  when  $f$  is harmonic.

In the context of biharmonic maps, Jiang [14] obtained the stress-energy tensor  $S_2$  given by

$$\begin{aligned} S_2(X, Y) = & \frac{1}{2} \|\tau(f)\|^2 \langle X, Y \rangle + \langle df, \nabla \tau(f) \rangle \langle X, Y \rangle \\ & - \langle X(f), \nabla_Y \tau(f) \rangle - \langle Y(f), \nabla_X \tau(f) \rangle, \end{aligned}$$

which satisfies

$$\operatorname{div} S_2 = -\langle \tau_2(f), df \rangle.$$

In the case of  $f : M^m \rightarrow N^n$  to be an isometric immersion, it follows that  $\operatorname{div} S = 0$ , since  $\tau(f)$  is normal to  $f$ . However, we have

$$\operatorname{div} S_2 = -\tau_2(f)^T,$$

and thus  $\operatorname{div} S_2$  does not always vanish.

**Definition 1.** An isometric immersion  $f : M^m \rightarrow N^n$  is called *biconservative* if its stress-energy tensor  $S_2$  is conservative, i.e.,  $\tau_2(f)^T = 0$ .

The following splitting result of the bitension field, with respect to its normal and tangent components, is well known (see, for example [7, 19, 20]).

**Proposition 2.1.** *Let  $f : M^m \rightarrow N^n$  be an isometric immersion between two Riemannian manifolds. Then  $f$  is biharmonic if and only if the tangent and normal components of  $\tau_2(f)$  vanish, i.e.,*

$$m \operatorname{grad} \|H\|^2 + 4 \operatorname{trace} A_{\nabla_{(\cdot)}^\perp H}(\cdot) + 4 \operatorname{trace} (\tilde{R}(\cdot, H) \cdot)^T = 0 \quad (2.1)$$

and

$$\operatorname{trace} \alpha_f(A_H(\cdot), H) - \Delta^\perp H + 2 \operatorname{trace} (\tilde{R}(\cdot, H) \cdot)^\perp = 0, \quad (2.2)$$

where  $\tilde{R}$  denotes the curvature tensor of  $N$ .

### 3 Basic facts about submanifolds in $\mathbb{Q}_\epsilon^n \times \mathbb{R}$

In order to study submanifolds  $f : M^m \rightarrow \mathbb{Q}_\epsilon^n \times \mathbb{R}$ , our approach is to regard  $f$  as an isometric immersion into  $\mathbb{E}^{n+2}$ , where  $\mathbb{E}^{n+2}$  denote either Euclidean space or Lorentzian space  $(n+2)$ -dimensional, according as  $\epsilon = 1$  or  $\epsilon = -1$ , respectively. Then we consider the canonical inclusion

$$i : \mathbb{Q}_\epsilon^n \times \mathbb{R} \rightarrow \mathbb{E}^{n+2}$$

and study the composition  $\tilde{f} = i \circ f$ . Notice that the vector field  $T$  is the gradient of the height function  $h = \langle \tilde{f}, i_* \partial_t \rangle$ .

Using that  $\partial_t$  is a parallel vector field in  $\mathbb{Q}_\epsilon^n \times \mathbb{R}$  we obtain, by differentiating (1.1), that

$$\nabla_X T = A_\eta X \quad (3.1)$$

and

$$\alpha_f(X, T) = -\nabla_X^\perp \eta, \quad (3.2)$$

for all  $X \in TM$ , where  $\alpha_f$  denotes the second fundamental form of  $f$  and  $A_\xi$  stands for the shape operator of  $f$  with respect to  $\xi \in TM^\perp$ , given by

$$\langle A_\xi X, Y \rangle = \langle \alpha_f(X, Y), \eta \rangle \text{ for all } X, Y \in TM.$$

The Gauss, Codazzi and Ricci equations for  $f$  are, respectively

$$\begin{aligned} R(X, Y)Z &= A_{\alpha(Y, Z)}X - A_{\alpha(X, Z)}Y + \epsilon(X \wedge Y \\ &\quad + \langle X, T \rangle Y \wedge T - \langle Y, T \rangle X \wedge T)Z, \end{aligned} \quad (3.3)$$

$$\left( \nabla_X^\perp \alpha \right) (Y, Z) - \left( \nabla_Y^\perp \alpha \right) (X, Z) = \epsilon \langle (X \wedge Y)T, Z \rangle \eta \quad (3.4)$$

and

$$R^\perp(X, Y)\xi = \alpha(X, A_\xi Y) - \alpha(A_\xi X, Y), \quad (3.5)$$

for all  $X, Y, Z \in TM$  and  $\xi \in TM^\perp$  (cf. [16] for more details).

In the case of hypersurfaces  $f : M^n \rightarrow \mathbb{Q}_\epsilon^n \times \mathbb{R}$ , the vector field  $\eta$  given in (1.1) can be written as

$$\eta = \nu N, \quad (3.6)$$

where  $N$  is a unit normal vector field along  $f$  and  $\nu$  is a smooth function on  $M$ . Thus the equations (3.1) and (3.2) become

$$\nabla_X T = \nu AX \quad \text{and} \quad X(\nu) = -\langle AX, T \rangle,$$

for all  $X \in TM$ , where  $A$  stands for the shape operator of  $f$  with respect to  $N$ .

### 3.1 The class $\mathcal{A}$

We will denote by  $\mathcal{A}$  the class of isometric immersions  $f : M^m \rightarrow \mathbb{Q}_\epsilon^n \times \mathbb{R}$  with the property that  $T$  is an eigenvector of all shape operators of  $f$ . This class was introduced in [21], where a complete description was given for hypersurfaces, and extended to submanifolds of  $\mathbb{Q}_\epsilon^n \times \mathbb{R}$  in [18]. Trivial examples are the slices  $\mathbb{Q}_\epsilon^n \times \{t\}$ , corresponding to the case in which  $T$

vanishes identically, and the vertical cylinders  $N^{m-1} \times \mathbb{R}$ , where  $N^{m-1}$  is a submanifold of  $\mathbb{Q}_\epsilon^n$ , which correspond to the case in which the normal vector field  $\eta$  vanishes identically.

Following the notation of [18], let us recall a way of construct more examples of submanifolds in this class. Let  $g : N^{m-1} \rightarrow \mathbb{Q}_\epsilon^n$  be an isometric immersion and suppose that there exists an orthonormal set of parallel normal vector fields  $\xi_1, \dots, \xi_k$  along  $g$ . Thus the vector subbundle  $E$  with rank  $k$  of  $TN^\perp$ , spanned by  $\xi_1, \dots, \xi_k$ , is parallel and flat. Let us denote by  $j : \mathbb{Q}_\epsilon^n \rightarrow \mathbb{Q}_\epsilon^n \times \mathbb{R}$  and  $i : \mathbb{Q}_\epsilon^n \times \mathbb{R} \rightarrow \mathbb{E}^{n+2}$  the canonical inclusions, and let  $l = i \circ j$ . Set

$$\tilde{\xi}_0 = l \circ g, \quad \tilde{\xi}_i = l_* \xi_i, \quad 1 \leq i \leq k, \quad \text{and} \quad \tilde{\xi}_{k+1} = i_* \partial_t.$$

Then the vector subbundle  $\tilde{E}$  of  $TN_{\tilde{g}}^\perp$ , where  $\tilde{g} = l \circ g$ , spanned by  $\tilde{\xi}_0, \dots, \tilde{\xi}_{k+1}$ , is parallel and flat, and we can define a vector bundle isometry

$$\phi : N^{m-1} \times \mathbb{E}^{k+2} \rightarrow \tilde{E}$$

by

$$\phi(x, y) = \sum_{i=0}^{k+1} y_i \tilde{\xi}_i(x),$$

for all  $x \in N^{m-1}$  and for all  $y = (y_0, \dots, y_{k+1}) \in \mathbb{E}^{k+2}$ . Using this isometry, we define a map  $f : N^{m-1} \times I \rightarrow \mathbb{Q}_\epsilon^n \times \mathbb{R}$  by

$$\tilde{f}(x, t) = (i \circ f)(x, t) = \phi(x, \alpha(t)), \quad (3.7)$$

where  $\alpha : I \rightarrow \mathbb{Q}^k \times \mathbb{R}$  is a regular curve with  $\sum_{i=0}^k \alpha_i^2 = 1$  and  $\alpha'_{k+1} \neq 0$ .

The main result concerning the map  $f$  given in (3.7) is that, at regular points,  $f$  is an isometric immersion in class  $\mathcal{A}$ . Conversely, given any isometric immersion  $f : M^m \rightarrow \mathbb{Q}_\epsilon^n \times \mathbb{R}$  in class  $\mathcal{A}$ , with  $m \geq 2$ ,  $f$  is locally given in this way (cf. [18, Theorem 2]). The map  $\tilde{f}$  is a partial tube over  $\tilde{g}$  with type fiber  $\alpha$  in the sense of [3]. Geometrically, the submanifold  $M^m = N^{m-1} \times I$  is obtained by the parallel transport of  $\alpha$  in a product submanifold  $\mathbb{Q}^k \times \mathbb{R}$  of a fixed normal space of  $\tilde{g}$  with respect to its normal connection.

We point out that, in the case of hypersurfaces,  $f$  is in class  $\mathcal{A}$  if and only if the vector field  $T$  in (1.1) is nowhere vanishing and  $\tilde{f}$  has flat normal bundle (cf. [21, Proposition 4]). Some important classes of hypersurfaces of  $\mathbb{Q}_\epsilon^n \times \mathbb{R}$  that are included in class  $\mathcal{A}$  are hypersurfaces with constant sectional curvature [17], rotational hypersurfaces [5] and constant angle hypersurfaces [21]. For submanifolds of higher codimension, we have that  $f$  is in class  $\mathcal{A}$  and it has flat normal bundle if and only if the vector field  $T$  in (1.1) is nowhere vanishing and  $\tilde{f}$  has flat normal bundle [18, Corollary 3].

## 4 Biconservative submanifolds in $\mathbb{Q}_\epsilon^n \times \mathbb{R}$

Let  $f : M^m \rightarrow \mathbb{Q}_\epsilon^n \times \mathbb{R}$  be an isometric immersion with nonzero parallel mean curvature vector field  $H$ . It follows from (2.1) and from the expression of the curvature tensor of  $\mathbb{Q}_\epsilon^n \times \mathbb{R}$  that  $f$  is biconservative if and only if

$$\epsilon \langle H, \eta \rangle T = 0, \quad (4.1)$$

where  $\eta$  and  $T$  denote the vector fields given in (1.1). Without loss of generality, we may assume that  $T$  and  $\eta$  are nowhere vanishing. Therefore, it follows from (4.1) that  $H$  is orthogonal to  $\partial_t$  and, thus

$$X \langle H, \partial_t \rangle = 0, \quad (4.2)$$

for all  $X \in TM$ . As  $\nabla_X^\perp H = 0$  and  $\tilde{\nabla}_X \partial_t = 0$ , it follows from (4.2) that

$$\langle A_H T, X \rangle = 0,$$

for all  $X \in TM$ , which implies that

$$A_H T = 0. \quad (4.3)$$

On the other hand, since  $H$  is parallel it follows from the Ricci equation that  $[A_H, A_\xi] = 0$  for every  $\xi \in TM_f^\perp$ . In particular, we have  $[A_H, A_\eta] = 0$ . Equivalently, the eigenspaces associated to  $A_H$  are invariant by  $A_\eta$ . In particular, if we denote by

$$E_0(H) = \{X \in TM : A_H X = 0\}, \quad (4.4)$$

we conclude that

$$A_\eta T \in E_0(H). \quad (4.5)$$

**Remark 4.1.** In the case of biconservative hypersurfaces with nonzero parallel mean curvature vector, the equation (4.1) can be written as

$$h\nu T = 0,$$

where  $\nu$  is the function given in (3.6) and  $h$  is the smooth function such that  $H = hN$ . Thus, as  $h \neq 0$ , a biconservative hypersurface  $f : M^n \rightarrow \mathbb{Q}_\epsilon^n \times \mathbb{R}$  with nonzero parallel mean curvature vector is either a slice  $\mathbb{Q}_\epsilon^n \times \{t\}$  or an open subset of a Riemannian product  $N^{n-1} \times \mathbb{R}$ , where  $N^{n-1}$  is a hypersurface of  $\mathbb{Q}_\epsilon^n$  with nonzero parallel mean curvature vector field.

Thus, by virtue of Remark 4.1, we will consider biconservative submanifolds with codimension greater than one.

#### 4.1 Biconservative submanifolds of co-dimension 2

Let us consider now the case of co-dimension 2, that is, a biconservative isometric immersion  $f : M^n \rightarrow \mathbb{Q}_\epsilon^{n+1} \times \mathbb{R}$  with nonzero parallel mean curvature vector field  $H$ . Let us consider the unit normal vector fields

$$\xi_1 = H/\|H\| \quad \text{and} \quad \xi_2 = \eta/\|\eta\|. \quad (4.6)$$

It follows from (4.1) that  $\{\xi_1, \xi_2\}$  is an orthonormal normal frame of  $f$ . Moreover, as  $\nabla^\perp \xi_1 = 0$  and  $f$  has co-dimension 2, we also have  $\nabla^\perp \xi_2 = 0$ .

Suppose first that the eigenspace  $E_0(H)$  given in (4.4) is one-dimensional, that is,  $E_0(H) = \text{span}\{T\}$ . This implies that  $A_\eta T = \lambda T$  for some smooth function  $\lambda$ . Thus, it follows from (3.2) that

$$-\nabla_X^\perp \eta = \frac{\lambda}{\|\eta\|^2} \langle T, X \rangle \eta.$$

In particular, we have  $\nabla_X^\perp \eta = 0$  for every  $X \in \{T\}^\perp$  and, from [18, Proposition 10], we conclude that  $f$  is in class  $\mathcal{A}$ .

**Remark 4.2.** *If  $E_0(H)$  is  $n$ -dimensional one has  $A_{\xi_1}$  identically zero. This implies that the mean curvature vector field  $H$  of  $f$  is a multiple of  $\xi_2$ , and this contradicts the fact that  $H$  and  $\eta$  are orthogonal, unless that  $\eta$  is identically zero.*

From now on, let us assume that  $\dim E_0(H) = k$ , with  $1 < k < n$ .

**Lemma 4.3.** *Let  $f : M^n \rightarrow \mathbb{Q}_\epsilon^{n+1} \times \mathbb{R}$  be a biconservative isometric immersion with nonzero parallel mean curvature vector field. Then there exists a local orthonormal frame  $X_1, \dots, X_n$  in  $M^n$ , with  $X_1 = T/\|T\|$ , such that:*

- (i) *The shape operators of  $f$  with respect to  $\xi_1$  and  $\xi_2$ , given in (4.6), have matrix representations given by*

$$A_{\xi_1} = \begin{pmatrix} 0 & 0 \\ 0 & S_1 \end{pmatrix} \quad \text{and} \quad A_{\xi_2} = \begin{pmatrix} S_2 & 0 \\ 0 & B \end{pmatrix}, \quad (4.7)$$

*where  $S_1$  and  $B$  are diagonalized matrices and  $S_2$  is a symmetric matrix such that  $\text{trace } S_1 = \text{const} \neq 0$ ,  $\text{trace } S_2 + \text{trace } B = 0$  and*

$$\text{trace } BS_1 = 0. \quad (4.8)$$

- (ii)  $\nabla_{X_i} X_j \in E_0(H)$ , for all  $1 \leq i, j \leq k$ .

*Proof.* Writing  $X_1 = T/\|T\|$ , consider the local orthonormal frame  $X_1, X_2, \dots, X_n$  in  $M^n$ , where  $X_2, \dots, X_n$  are eigenvectors of  $A_{\xi_1}$  such that

$$E_0(H) = \text{span}\{X_1, X_2, \dots, X_k\}.$$



Thus we have the first equation of (4.7). Moreover, since  $\xi_1$  is proportional to  $H$  and  $H$  has constant length, we have  $\text{trace } A_{\xi_1} = \text{trace } S_1 = \text{const} \neq 0$  and  $\text{trace } A_{\xi_2} = 0$ . On the other hand, by a simple computation, one can see that the Ricci equation  $R^\perp(X_i, X_j)\xi_1 = 0$  takes the form

$$\alpha_f(X_i, A_{\xi_1} X_j) - \alpha_f(A_{\xi_1} X_i, X_j) = 0. \quad (4.9)$$

For  $1 \leq i \leq k$  and  $k+1 \leq j \leq n$ , the equation (4.9) gives

$$\alpha_f(X_i, X_j) = 0. \quad (4.10)$$

Therefore, the matrix representation of  $A_{\xi_2}$  takes the form given in the second equation of (4.7). Moreover, for  $k+1 \leq i \neq j \leq n$ , (4.9) becomes

$$(\lambda_i - \lambda_j)\alpha_f(X_i, X_j) = 0.$$

Now, if the distribution

$$\Gamma_i = \{X \in TM : A_{\xi_1} X = \lambda_i X\}$$

has dimension  $m_i > 1$ , then we have  $A_{\xi_1}|_{\Gamma_i} = \lambda_i \text{Id}$  and, by replacing indices if necessary, we may assume that  $\Gamma_i = \text{span}\{X_i, X_{i+1}, \dots, X_{i+m_i-1}\}$ . Therefore, by redefining  $X_i, X_{i+1}, \dots, X_{i+m_i-1}$  properly, we may diagonalize  $A_{\xi_2}|_{\Gamma_1}$ . Since  $A_{\xi_1}|_{\Gamma_1}$  is proportional to identity matrix it, no matter, remains diagonalized. Summing up, we see that, by redefining  $X_{k+1}, X_{k+2}, \dots, X_n$  properly, one can diagonalize the matrix  $B$ . Then, we can write

$$S_1 = \text{diag}(\lambda_{k+1}, \lambda_{k+2}, \dots, \lambda_n) \quad \text{and} \quad B = \text{diag}(\mu_{k+1}, \mu_{k+2}, \dots, \mu_n)$$

for some smooth functions  $\lambda_i, \mu_i$ , with  $k+1 \leq i \leq n$ . In order to obtain (4.8), we need to show that

$$\sum_{i=k+1}^n \lambda_i \mu_i = 0. \quad (4.11)$$

By a direct computation, it follows from the Codazzi equation that

$$X_1(\lambda_i) + \langle \nabla_{X_i} X_1, X_i \rangle \lambda_i = 0, \quad (4.12)$$

for  $k+1 \leq i \leq n$ . On the other hand, from (1.1) we have

$$\partial_t = \cos \theta X_1 + \sin \theta \xi_2 \quad (4.13)$$

for a smooth function  $\theta \neq \frac{\pi}{2}$ . Since  $\partial_t$  is parallel, equation (4.13) yields

$$0 = \cos \theta \langle \nabla_{X_i} X_1, X_i \rangle - \sin \theta \langle A_{\xi_2} X_i, X_i \rangle. \quad (4.14)$$

Combining (4.12) and (4.14), we get

$$X_1(\lambda_i) = \tan \theta \lambda_i \mu_i.$$

By summing this equation on  $i$  and taking into account

$$\text{trace } S_1 = \sum_{i=k+1}^n \lambda_i = \text{const},$$

we get (4.11), which proves the assertion in (i). Finally, for  $1 \leq i, l \leq k$  and  $k+1 \leq j \leq n$ , we obtain from Codazzi equation that

$$\langle \tilde{R}(X_i, X_j)X_l, \xi_1 \rangle = 0.$$

Then, using (4.10), we obtain

$$\langle \nabla_{X_i} X_l, X_j \rangle = 0,$$

for all  $1 \leq i, l \leq k$  and  $k+1 \leq j \leq n$ , and this proves (ii).  $\square$

**Corollary 4.4.** *Let  $f : M^n \rightarrow \mathbb{Q}_\epsilon^{n+1} \times \mathbb{R}$  be a biconservative isometric immersion with nonzero parallel mean curvature vector field. Then  $E_0(H)$  is an involutive distribution.*

*Proof.* It is clear when  $\dim E_0(H) = 1$ . If  $\dim E_0(H) > 1$ , consider a local orthonormal frame  $X_1, \dots, X_n$  in  $M^n$  constructed in Lemma 4.3. From condition (ii), we have  $[X, Y] \in E_0(H)$ , for all  $X, Y \in E_0(H)$ , which completes the proof.  $\square$

In the next result we obtain a necessary and sufficient condition for a biconservative submanifold with parallel mean curvature vector to be biharmonic.

**Proposition 4.5.** *Let  $f : M^n \rightarrow \mathbb{Q}_\epsilon^{n+1} \times \mathbb{R}$  be a biconservative isometric immersion with nonzero parallel mean curvature vector field. Then,  $M$  is biharmonic if and only if the equation*

$$\text{trace } A_{\xi_1}^2 + \|T\|^2 = n \tag{4.15}$$

*is satisfied, where  $\xi_1$  is the unit normal vector field given in (4.6).*

*Proof.* By Proposition 2.1,  $M$  is biharmonic if and only if the equation (2.2) is satisfied. Consider the local orthonormal frame  $\{X_1, \dots, X_n\}$  given in Lemma 4.3. Since, the mean curvature vector field  $H$  is parallel and  $\langle H, \eta \rangle = 0$ , the equation (2.2) turns into (4.15) by virtue of (4.8).  $\square$

## 5 Biconservative submanifolds in $\mathbb{Q}_\epsilon^4 \times \mathbb{R}$

In this section we prove Theorem 1.1 in two steps. In the first one, we prove that there is an explicit way to construct 3-dimensional biconservative submanifolds in  $\mathbb{Q}_\epsilon^4 \times \mathbb{R}$  with parallel mean curvature vector field. In the second step, we prove that any 3-dimensional biconservative submanifolds in  $\mathbb{Q}_\epsilon^4 \times \mathbb{R}$ , with nonzero parallel mean curvature vector field, is locally given as in the previous construction.

### 5.1 Examples of biconservative submanifolds

Here we prove the first part of Theorem 1.1.

**Theorem 5.1.** *Let  $\phi : M^2 \rightarrow \mathbb{Q}_a^2 \times \mathbb{R}$  be an oriented minimal surface such that the vector field  $T_\phi$  defined by (1.1) is nowhere vanishing, where  $a \neq 0$  and  $|a| < 1$ . Let  $b > 0$  be a real number such that  $a^2 + b^2 = 1$ . Let now*

$$f : M^3 = M^2 \times I \rightarrow \mathbb{Q}_\epsilon^4 \times \mathbb{R}$$

be given by

$$f(p, s) = \left( b \cos \frac{s}{b}, b \sin \frac{s}{b}, \phi(p) \right). \quad (5.1)$$

Then the map  $f$  defines, at regular points, an isometric immersion with  $\langle H, \eta \rangle = 0$ . Moreover,  $f$  is a biconservative isometric immersion with parallel mean curvature vector field if and only if  $\phi$  is a vertical cylinder.

*Proof.* Let  $\{X_1, X_2, X_3\}$  be a local orthonormal tangent frame of  $M^3$ , with  $X_3 = \partial_s$ . By putting  $Y_1 = \pi_* X_1$  and  $Y_2 = \pi_* X_2$ , where  $\pi : M^3 \rightarrow M^2$  denotes the canonical projection,  $\pi(p, s) = p$ , we get that  $\{Y_1, Y_2\}$  is a local orthonormal tangent frame of  $M^2$ . If  $N = (N_1, N_2, N_3, N_4)$  denotes the unit normal vector field of  $M^2$  in  $\mathbb{Q}_a^2 \times \mathbb{R}$ , then

$$\xi_1 = \left( -a \cos \frac{s}{b}, -a \sin \frac{s}{b}, \frac{b}{a} (\pi_1 \circ \phi) \right) \quad \text{and} \quad \xi_2 = (0, 0, N)$$

provides a local orthonormal normal frame of  $f$  in  $\mathbb{Q}_\epsilon^4 \times \mathbb{R} \subset \mathbb{E}^6$ , where  $\pi_1 : \mathbb{Q}_a^2 \times \mathbb{R} \rightarrow \mathbb{Q}_a^2$  denotes the canonical projection. Note that we have

$$\langle \xi_1, \partial_t \rangle = 0.$$

In terms of the tangent frame  $\{Y_1, Y_2\}$  of  $M^2$ , the shape operator  $A_N$  is given by

$$A_N = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & -a_{11} \end{pmatrix},$$

for some smooth functions  $a_{11}$  and  $a_{12}$ . By a direct computation, one can see that the matrix representation of  $A_{\xi_2}$ , with respect to  $\{X_1, X_2, X_3\}$ , take the form

$$A_{\xi_2} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{12} & -a_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (5.2)$$

It follows from (5.2) that  $H = c \cdot \xi_1$ , where  $c = \langle H, \xi_1 \rangle$ , which implies  $\langle H, \eta \rangle = 0$ . Moreover, we have  $T_\phi = \pi_* T_f$ , since  $\langle \partial_t, \partial_s \rangle = 0$ . Thus, as  $T_\phi$  is nowhere vanishing, and therefore also  $T_f$ , it is straightforward to verify that  $H$  is parallel if and only if  $N_4 = 0$ . It means that  $\partial_t$  is orthogonal to  $M^2$ , which implies that  $\|T_\phi\| = 1$ . Thus,  $M^2$  is a vertical cylinder  $M^2 = \gamma \times \mathbb{R}$  over a geodesic curve  $\gamma$  in  $\mathbb{Q}_a^2$ .  $\square$

**Corollary 5.2.** *If  $f : M^3 \rightarrow \mathbb{Q}_\epsilon^4 \times \mathbb{R}$  is a biconservative submanifold with nonzero parallel mean curvature vector field, locally given as in (5.1), then  $f$  is an immersion in class  $\mathcal{A}$ .*

*Proof.* As  $f$  is locally given as in (5.1) it follows, in particular, that  $\phi$  is in class  $\mathcal{A}$ . Thus, the vector field  $T_\phi$  associated to  $\phi$ , given in (1.1), is a principal direction of  $\phi$ . This implies that

$$\langle A_\zeta T_\phi, Z \rangle = 0,$$

for all  $\zeta \in TM_\phi^\perp$ , where  $Z$  is tangent to  $\phi$  and orthogonal to  $T_\phi$ . With the notations as in Theorem 5.1, and by considering

$$Y_1 = \frac{T_\phi}{\|T_\phi\|} \quad \text{and} \quad Y_2 = \frac{Z}{\|Z\|},$$

we have

$$A_N = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \quad \text{and} \quad A_{\xi_2} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This shows that  $T_f$  is an eigenvector of  $A_{\xi_2}$ , since  $T_\phi = \pi_* T_f$ .  $\square$

## 5.2 Classification results in $\mathbb{Q}_\epsilon^4 \times \mathbb{R}$

Finally, in this subsection, we prove the converse of Theorem 1.1. Here we will consider biconservative isometric immersion  $f : M^3 \rightarrow \mathbb{Q}_\epsilon^4 \times \mathbb{R}$ , with nonzero parallel mean curvature vector field  $H$  such that  $\dim E_0(H) = 2$ . Let us consider the local orthonormal frame  $\{X_1, X_2, X_3\}$  given in Lemma 4.3. Denoting by  $\xi_1$  and  $\xi_2$  as in (4.6), we have

$$A_{\xi_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3\|H\| \end{pmatrix} \quad (5.3)$$

and

$$A_{\xi_2} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}, \quad (5.4)$$

for some smooth functions  $a_{11}$ ,  $a_{22}$  and  $a_{33}$ , with  $a_{11} + a_{22} + a_{33} = 0$ . Note that, from (4.8), we have  $a_{33} = 0$  and thus, (5.4) becomes (5.2).

On the other hand, we can write the vector field  $\partial_t$  as

$$\partial_t = \cos \theta X_1 + \sin \theta \xi_2, \quad (5.5)$$

for a smooth function  $\theta \neq \frac{\pi}{2}$ . Applying  $X_3$  to (5.5), we obtain

$$\nabla_{X_3} X_1 = 0.$$

Moreover, from the Codazzi equation, we obtain  $\langle \tilde{R}(X_2, X_3)X_3, \xi_1 \rangle = 0$ , that implies

$$\langle \nabla_{X_3} X_2, X_3 \rangle = 0. \quad (5.6)$$

By putting  $\langle \nabla_{X_i} X_1, X_2 \rangle = \phi_i$ , for  $1 \leq i \leq 2$ , we have the following:

**Lemma 5.3.** *In terms of the local orthonormal frame  $\{X_1, X_2, X_3\}$  in  $M^3$ , the Levi-Civita connection of  $M^3$  is given by*

$$\begin{aligned} \nabla_{X_1} X_1 &= \phi_1 X_2, & \nabla_{X_1} X_2 &= -\phi_1 X_1, & \nabla_{X_1} X_3 &= 0, \\ \nabla_{X_2} X_1 &= \phi_2 X_2, & \nabla_{X_2} X_2 &= -\phi_2 X_1, & \nabla_{X_2} X_3 &= 0, \\ \nabla_{X_3} X_1 &= 0, & \nabla_{X_3} X_2 &= 0, & \nabla_{X_3} X_3 &= 0. \end{aligned} \quad (5.7)$$

*Proof.* A straightforward computation.  $\square$

**Lemma 5.4.** *There exists a local coordinate system  $(u_1, u_2, s)$  in  $M^3$  such that  $E_0(H) = \text{span}\{\partial_{u_1}, \partial_{u_2}\}$ ,  $X_3 = \partial_s$  and  $f$  decomposes as*

$$f(u_1, u_2, s) = \Gamma_1(s) + \Gamma_2(u_1, u_2), \quad (5.8)$$

for some smooth functions  $\Gamma_1$  and  $\Gamma_2$ . Moreover,  $f(u_1, u_2, \cdot)$  are the integral curves of  $X_3$  for any  $(u_1, u_2)$  and  $f(\cdot, \cdot, s)$  are the integral submanifolds of  $E_0(H)$  for any  $s$ .

*Proof.* By Corollary 4.4, the tangent bundle  $TM$  decomposes orthogonally as

$$TM = E_0(H) \oplus (E_0(H))^\perp.$$

Therefore, there exists a local coordinate system  $(u_1, u_2, s)$  in  $M^3$  such that

$$E_0(H) = \text{span}\{\partial_{u_1}, \partial_{u_2}\} \quad \text{and} \quad (E_0(H))^\perp = \text{span}\{\partial_s\}$$

(see [15, p. 182]). Thus  $X_3 = E\partial_s$  for some smooth function  $E$  on  $M^3$ . On the other hand, since  $[\partial_{u_i}, \partial_s] = 0$ , for  $1 \leq i \leq 2$ , we have  $\nabla_{\partial_{u_i}} \partial_s = \nabla_{\partial_s} \partial_{u_i}$ . However, by considering (5.7), one can see that

$$\nabla_{\partial_{u_i}} \partial_s \in E_0(H)^\perp \quad \text{and} \quad \nabla_{\partial_s} \partial_{u_i} \in E_0(H), \quad (5.9)$$

for  $1 \leq i \leq 2$ . By considering (5.9) and (5.7), and taking into account the fact that  $\alpha_{\tilde{f}}(X, Y) = \alpha(X, Y)$ , whenever  $X, Y$  are orthogonal tangent vector fields on  $M^3$ , we obtain

$$\hat{\nabla}_{\partial_{u_i}} \partial_s = \hat{\nabla}_{\partial_s} \partial_{u_i} = 0 \quad (5.10)$$

and

$$\partial_{u_i}(E) = 0, \quad (5.11)$$

for all  $1 \leq i \leq 2$ . From (5.10), we obtain (5.8) for some smooth functions  $\Gamma_1$  and  $\Gamma_2$ . Moreover, equation (5.11) implies that  $E = E(s)$ . Therefore, by re-defining the parameter  $s$  properly, we may assume that  $E = 1$ , which concludes the proof.  $\square$

**Proposition 5.5.** *Let  $f : M^3 \rightarrow \mathbb{Q}_\epsilon^4 \times \mathbb{R}$  be a biconservative isometric immersion with nonzero parallel mean curvature vector field  $H$ . Suppose that  $\dim E_0(H) = 2$  and let  $p \in M$ . Then the following assertions hold:*

- (i) *An integral submanifold  $N$  of  $E_0(H)$  through  $p$  lies on a 4-plane  $\Pi_1$  of  $\mathbb{E}^6$  containing the factor  $\partial_t$ . Moreover,  $N$  is congruent to a minimal surface  $\phi : M^2 \rightarrow \mathbb{Q}_a^2 \times \mathbb{R}$ .*
- (ii) *An integral curve of  $X_3$  through  $p$  is an open subset of a circle of radius  $b = \frac{1}{\sqrt{c^2+1}}$  contained on a 2-plane  $\Pi_2$  of  $\mathbb{E}^6$ , where  $c = 3\|H\|$ .*

*Proof.* Let  $N$  be an integral submanifold of  $E_0(H)$  through  $p$ . Define vector fields  $\zeta_1, \dots, \zeta_6$  along  $N$  by

$$\zeta_i = X_i|_N \quad \text{and} \quad \zeta_j = \xi_j|_N,$$

for  $1 \leq i \leq 3$  and  $1 \leq j \leq 3$ , where  $\xi_3$  is the restriction of the unit normal vector field of the immersion  $f : M^3 \rightarrow \mathbb{Q}_\epsilon^4 \times \mathbb{R}$  to  $M^3$ . Note that  $\zeta_1, \zeta_2$  span  $TN$ , while the vector fields  $\zeta_3, \dots, \zeta_6$  span the normal bundle  $TN^\perp$  in  $\mathbb{E}^6$ . By taking into account the fact that  $\alpha_{\tilde{f}}(X, Y) = \alpha_f(X, Y)$ , whenever  $X, Y$  are orthogonal tangent vector fields on  $M^3$ , and considering (5.2), (5.3) and Lemma 5.3, we get

$$\hat{\nabla}_X \zeta_3 = \hat{\nabla}_X \zeta_4 = 0,$$

for all  $X \in TM$ , where  $\hat{\nabla}$  is the Levi-Civita connection of  $\mathbb{E}^6$ . This yields that  $N$  lies on a 4-plane  $\Pi_1$  on which  $\partial_t$  lies. Moreover, the unit normal

vector field of  $N$  in  $\Pi_1 \cap (\mathbb{Q}_\epsilon^4 \times \mathbb{R}) \cong \mathbb{Q}_a^2 \times \mathbb{R}$  is  $\zeta_5$ , and the shape operator of  $N$  along  $\zeta_5$  becomes

$$A_{\zeta_5} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & -a_{12} \end{pmatrix},$$

which shows that  $N$  is congruent to a minimal surface in  $\mathbb{Q}_a^2 \times \mathbb{R}$ . This proves the assertion (i). In order to prove (ii), let us consider an integral curve  $\gamma$  of  $X_3$  through  $p$  and define

$$\zeta = \frac{c}{\sqrt{c^2 + 1}}\zeta_1 - \frac{1}{\sqrt{c^2 + 1}}\zeta_3$$

as a vector field along  $\gamma$ . Then we have

$$\hat{\nabla}_{\gamma'}\gamma' = \sqrt{c^2 + 1}\zeta \quad \text{and} \quad \hat{\nabla}_{\gamma'}\zeta = -\sqrt{c^2 + 1}\gamma'.$$

Thus  $\gamma$  is an open subset of a circle lying on the 2-plane  $\Pi_2$  spanned by  $\gamma'$  and  $\zeta$ . This proves (ii) and concludes the proof.  $\square$

By summing up Lemma 5.4 and Proposition 5.5, we get the converse of Theorem 1.1, which can be stated as follow.

**Theorem 5.6.** *Let  $f : M^3 \rightarrow \mathbb{Q}_\epsilon^4 \times \mathbb{R}$  be a biconservative isometric immersion with nonzero parallel mean curvature vector field  $H$ . Then  $f$  is either an open subset of a slice  $\mathbb{Q}_\epsilon^4 \times \{t_0\}$  for some  $t_0 \in \mathbb{R}$ , an open subset of a Riemannian product  $N^3 \times \mathbb{R}$ , where  $N^3$  is a hypersurface of  $\mathbb{Q}_\epsilon^4$ , or it is locally congruent to the immersion  $f$  described in Theorem 5.1. In particular,  $f$  belongs to class  $\mathcal{A}$ .*

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Fernando Manfio – Universidade de São Paulo, Brazil

*E-mail address:* `manfio@icmc.usp.br`

Nurettin Cenk Turgay – Istanbul Technical University, Turkey

*E-mail address:* `turgayn@itu.edu.tr`

Abhitosh Upadhyay – Harish Chandra Research Institute, India

*E-mail address:* `abhi.basti.ipu@gmail.com`, `abhitoshupadhyay@hri.res.in`